Geometry of Polynomials : Chapter 2

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1 Alon-Bopanna Bound

Now, we will introduce a bound on the 2^{nd} smallest eigenvalue of L in terms of the maximum degree of G and its diameter. We first start by defining the distance in a graph between two edges e and f by the minimum number of edges in a path that connects an end point of e and an end point of f. This section is largely adapted from [14]

Theorem 1.1 (Alon-Bopanna bound)**.** *[14] Let G be a graph in which there are two edges of distance at least $2k+2$, and d be the maximum degree of G, λ_2 be the 2^{nd} smallest eigenvalue of the Laplacian L. Then*

$$\lambda_2 \leq d - 2\sqrt{d-1} + (2\sqrt{d-1} - 1)/(k+1)$$

Proof. Let $G = (V, E)$ be the given graph, with maximum degree d. Let v_1, v_2 and u_1, u_2 be two edges of distance at least $2k+2$ between them. Let $V_0 = \{v_1, v_2\}$ and $U_0 = \{u_1, u_2\}$. Let V_i be the set of all vertices of distance i from V_0, $i = 1, ..., k$. Similarly, Let U_i be the set of all vertices of distance i from U_0.

Observe that the union of V_i is disjoint from the union of U_j, and there are no edges that connect a vertex in the first union to a vertex in the second. Moreover, $|V_i| \leq (d-1)|V_{i-1}|$ for all $i = 1, ..., k$. Similarly, $|U_i| \leq (d-1)|U_{i-1}|$ for all $i = 1, ..., k$.

Next, let $a, b \in \mathbb{R}$ such that $a > 0, b < 0$. Let $f : V \mapsto \mathbb{R}$ be defined by

$$f(v) \stackrel{\text{def}}{=} \begin{cases} a(d-1)^{-i/2} & \text{if } v \in V_i, 0 \leq i \leq k \\ b(d-1)^{-i/2} & \text{if } v \in U_i, 0 \leq i \leq k \\ 0 & \text{otherwise} \end{cases}$$

Note that a and b can be chosen such that $\sum_{v \in V} f(v) = 0$. Let a, b be chosen as such. By variational characterization of eigenvalues, $(f^T L f)/(f^T f) \geq \lambda_2$, taken over all nonzero functions f satisfying $\sum_{v \in V} f(v) = 0$. Observe that

$f^T f = A_1 + B_1$, where

$$A_1 = a^2 \sum_{i=0}^{k} \frac{|V_i|}{(d-1)^i}$$

$$B_1 = b^2 \sum_{j=0}^{k} \frac{|U_j|}{(d-1)^j}$$

Since $f^T L f = \sum_{\{u,v\}\in E}(f(u)-f(v))^2$, and since there are no edges joining a vertex in V_i to a vertex in U_j and there are most $d-1$ edges joining a vertex of V_i to a vertex of V_{i+1}, similarly for U_j and U_{j+1}, we have $f^T L f = A_2 + B_2$, where

$$A_2 \le a^2 \left(\sum_{i=0}^{k-1} |V_i|(d-1) \left(\frac{1}{(d-1)^{i/2}} - \frac{1}{(d-1)^{(i+1)/2}} \right)^2 + |V_k| \frac{d-1}{(d-1)^k} \right)$$

$$B_2 \le b^2 \left(\sum_{j=0}^{k-1} |U_j|(d-1) \left(\frac{1}{(d-1)^{j/2}} - \frac{1}{(d-1)^{(j+1)/2}} \right)^2 + |U_k| \frac{d-1}{(d-1)^k} \right)$$

To establish an upper bound for $\frac{A_2+B_2}{A_1+B_1}$, we observe that if a number M is an upper bound for both $\frac{A_2}{A_1}$ and $\frac{B_2}{B_1}$, then M will be an upper bound for $\frac{A_2+B_2}{A_1+B_1}$. Continuing from the upper bound for A_2, we have

$$\begin{aligned}
A_2 &\le a^2 (\sum_{i=0}^{k-1} |V_i|(d-1) \left(\frac{1}{(d-1)^{i/2}} - \frac{1}{(d-1)^{(i+1)/2}} \right)^2 + |V_k| \frac{d-1}{(d-1)^k}) \\
&= a^2 (\sum_{i=0}^{k-1} |V_i|(d-1) \left(\frac{1}{(d-1)^{i}} - \frac{2}{(d-1)^{(i+\frac{1}{2})}} + \frac{1}{(d-1)^{(i+1)}} \right) \\
&+ ((d-2\sqrt{d-1}) + (2\sqrt{d-1}-1)) \frac{|V_k|}{(d-1)^k}) \\
&= a^2 (\sum_{i=0}^{k-1} \frac{|V_i|}{(d-1)^i} (d-2\sqrt{d-1}) \\
&+ ((d-2\sqrt{d-1}) + (2\sqrt{d-1}-1)) \frac{|V_k|}{(d-1)^k}) \\
&= a^2 (\sum_{i=0}^{k} \frac{|V_i|}{(d-1)^i} (d-2\sqrt{d-1}) + (2\sqrt{d-1}-1) \frac{|V_k|}{(d-1)^k}) \\
&\le (d-2\sqrt{d-1}) A_1 + (2\sqrt{d-1}-1) \frac{A_1}{k+1}
\end{aligned}$$

where the last inequality holds because $\frac{V_i}{(d-1)^i}$ is a non-increasing sequence. Thus,

$$\frac{A_2}{A_1} \le d - 2\sqrt{d-1} + (2\sqrt{d-1}-1)\frac{1}{k+1}$$

The same bound holds for $\frac{B_2}{B_1}$ by repeating the proof, and thus it is also the upper bound for λ_2. □

Note that in a $d-$regular graph, the eigenvalue of the adjacency matrix A is related to the eigenvalues of L. Let $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ be the eigenvalues of L, and $\mu_1 \geq \mu_2 \geq ... \geq \mu_n$ be the eigenvalues of A. Since G is a $d-$regular graph, $D = Id$. Hence

$$L = Id - A$$

Hence

$$\lambda_i = d - \mu_i$$

Therefore, Theorem 1.1 implies in a $d - regular$ graph G, that the 2^{nd} largest eigenvalue of the adjacency matrix A of G containing two edges where the distance between which is at least $2k + 2$ is at least

$$2\sqrt{d-1}\left(1 - \frac{1}{k+1}\right) + \frac{1}{k+1}$$

2 Ramanujan Graph

In this subsection, we will be relating the Alon-Bopanna bound to ramanujan graphs. This will largely be adapted from [13]. Ramanujan graphs are defined in terms of the eigenvalues of their adjacency matrices. If G is a $d-$regular graph and A is its adjacency matrix, then d is always an eigenvalue of A. The matrix A has an eigenvalue of $-d$ if and only if G is bipartite. The eigenvalues d and $-d$ when G is bipartite are called the trivial eigenvalues of A. We say a $d-$regular graph is Ramanujan if all of its non-trivial eigenvalues lie between $-2\sqrt{d-1}$ and $2\sqrt{d-1}$.

Bilu and Linial [1] suggested the construction of Ramanujan graphs through a sequence of 2-lifts of a base graph.The 2-lift of $G = (V, E)$ is a graph that has two vertices for each vertex in V. The pair of vertices is called the fibre of the original vertex. Every edge in E corresponds to the two edges in 2-lift. If (u, v) is an edge in $E, \{u_0, u_1\}$ is the fibre of u, and $\{v_0, v_1\}$ is the fibre of v, then the 2-lift can contain either of the following pair of edges

$$\{(u_0, v_0), (u_1, v_1)\} \tag{1}$$

$$\{(u_0, v_1), (u_1, v_0)\} \tag{2}$$

If only (1) edge pair type appear, then the 2-lift is just two disjoint copies of the original graph. If only (2) edge pair type appears, then we obtain the double-cover of G.

To analyze the eigenvalues of a 2-lift, Bilu and Linial study signings $s : E \rightarrow \{\pm 1\}$ of the edges of G. They place signings in one-to-one correspondence with 2-lifts by setting $s(u, v) = 1$ if edges of type (1) appear in the 2-lift, and $s(u, v) =$

-1 if edges of type (2) appear. We then define the signed adjacency matrix A_s of G, similar to the adjacency matrix, except that the entries corresponding to an edge (u, v) is $s(u, v)$. They proved that the eigenvalues of the 2-lift are the union, taken with multiplicity, of the eigenvalues of A and A_s.

We say that a bipartite graph is (c, d)-biregular if all vertices on one side of the bipartition have degree c and all vertices on the other side have degree d. The adjacency matrix of a (c, d) biregular graph always has eigenvalues $\pm\sqrt{cd}$; these are its trivial eigenvalues. Feng and Li [5] proved a generalization of the Alon-Boppana bound that applies to (c, d)-biregular graphs: for all $\epsilon > 0$, all sufficiently large (c, d)-biregular graphs have a non-trivial eigenvalue that is at least $\sqrt{c-1} + \sqrt{d-1} - \epsilon$. Thus, we say that a (c, d)-biregular graph is Ramanujan if all of its non-trivial eigenvalues have absolute value at most $\sqrt{c-1} + \sqrt{d-1}$. We prove the existence of infinite families of (c, d)-biregular Ramanujan graphs for all $c, d \geq 3$.

The regular and biregular Ramanujan graphs discussed above are actually special cases of a more general phenomenon. To describe it, we will require a construction known as the universal cover. The universal cover of a graph G is the infinite tree T such that every connected lift of G is a quotient of the tree. It can be defined concretely by first fixing a "root" vertex $v_0 \in G$, and then placing one vertex in T for every non-backtracking walk $(v_0, v_1, \ldots, v_\ell)$ of any length $\ell \in \mathbb{N}$ starting at v_0, where a walk is non-backtracking if $v_{i-1} \neq v_{i+1}$ for all i. Two vertices of T are adjacent if and only if the walk corresponding to one can be obtained by appending one vertex to the walk corresponding to the other. That is, the edges of T are all of the form $(v_0, v_1, \ldots, v_\ell) \sim (v_0, v_1, \ldots, v_\ell, v_{\ell+1})$. The universal cover of a graph is unique up to isomorphism, independent of the choice of v_0.

The adjacency matrix A_T of the universal cover T is an infinite-dimensional symmetric matrix. We will be interested in the spectral radius $\rho(T)$ of T, which may be defined as:

$$\rho(T) := \sup_{\|x\|_2 = 1} \|A_T x\|_2$$

where $\|x\|_2^2 := \sum_{i=1}^{\infty} x(i)^2$ whenever the series converges. Naturally, the spectral radius of a finite tree is defined to be the norm of its adjacency matrix.

With these notions in hand, we can state the definition of an irregular Ramanujan graph. As before, the largest (and smallest, in the bipartite case) eigenvalues of finite adjacency matrices are considered trivial. Greenberg [9] showed that for every $\epsilon > 0$ and every infinite family of graphs that have the same universal cover T, all sufficiently large graphs in the family have a non-trivial eigenvalue that is at least $\rho(T) - \epsilon$. Following Hoory, Linial, and Wigderson [11], we therefore define an arbitrary graph to be Ramanujan if all of its non-trivial eigenvalues are smaller in absolute value than the spectral radius of its universal cover.

The universal cover of every d-regular graph is the infinite d-ary tree, whereas the universal cover of every (c, d)-biregular graph is the infinite (c, d)-biregular tree in which the degrees alternate between c and d on every other level [12]. The former tree is known to have spectral radius $2\sqrt{d-1}$ while the latter has a spectral radius of $\sqrt{c-1}+\sqrt{d-1}$. Thus, a definition based on universal covers generalizes both the regular and biregular definitions of Ramanujan graphs, and the bound of Greenberg generalizes both the Alon-Boppana and Feng-Li bounds.

In this general setting, we show that every graph G has a 2-lift in which all of the new eigenvalues are less than the spectral radius of its universal cover. Applying these 2-lifts inductively to any finite irregular bipartite Ramanujan graph yields an infinite family of irregular bipartite Ramanujan graphs whose degree distribution matches that of the initial graph (since taking a 2-lift simply doubles the number of vertices of each degree). In particular, applying them to the (c, d)-biregular complete bipartite graph yields an infinite family of (c, d)-biregular Ramanujan graphs. As far as we know, infinite families of irregular Ramanujan graphs were not known to exist prior to this work.

A covering graph $\tilde{G}$ of a graph G is a graph where there exist a surjective map $f : \tilde{G} \to G$ that is a local isomorphism. This means that for every vertex $v \in \tilde{G}$, the neighbourhood of v is mapped bijectively onto a neighbourhood of $f(v)$ in G.

A universal cover of a graph is a specific covering graph that could be constructed from G. If G is a tree, then G is a universal cover of itself. Otherwise for any finite connected graph G, the universal cover can be constructed to be an infinite tree.

2.1 2-Lifts and The Matching Polynomial

For a graph G, a matching in a graph G is a subset of edges of which no two edges have a vertex in common. Let m_i denote the number of matchings in G with i edges. Set $m_0 = 1$. Heilmann and Lieb [10] defined the matching polynomial of G to be the polynomial

$$\mu_G(x) \stackrel{\text{def}}{=} \sum_{i \geq 0} x^{n-2i}(-1)^i m_i$$

where n is the number of vertices in the graph.

They proved the following two theorems about the matching polynomial that we will exploit here.

Theorem 2.1. *[10] For every graph G, $\mu_G(x)$ has only real roots*

Theorem 2.2. *[10] For every graph G of maximum degree d, all of the roots of $\mu_G(x)$ have absolute value of at most $2\sqrt{d-1}$.*

The two theorems will allow us to prove the existence of infinite families of $d-$regular bipartite Ramanujan graphs. To handle the irregular case, we will require a refinement of those results due to Godsil[6, 7, 8]. This refinement uses the concept of a path tree, as defined by Godsil [7]. The definition follows below:

Definition 2.3. *Given a graph G and a vertex u, the path tree $P(G,u)$ contains one vertex for every path in G (with distinct vertices) that starts at u. Two paths are adjacent if one can be obtained by appending one vertex to the other. That is, all edges of $P(G,u)$ are all of the form $(u, v_1, ..., v_\ell) \sim (u, v_1, ..., v_\ell, v_{\ell+1})$.*

Note that a path in G is a walk that does not visit any vertex twice. The path tree provides a natural relationship between the roots of the matching polynomial of a graph and the spectral radius of its universal cover:

Theorem 2.4. *[7] Let $P(G,u)$ be a path tree of G. Then the matching polynomial of G divides the characteristic polynomial of the adjacency matrix of $P(G,u)$. In particular, all of the roots of $\mu_G(x)$ are real and have absolute value at most $\rho(P(G,u))$.*

Lemma 2.5. *Let G be a graph and let T be its universal cover. Then the roots of $\mu_G(x)$ are bounded in absolute value by $\rho(T)$.*

Proof. Let u be any vertex of G and let P be the path tree rooted at u. Since the paths that correspond to the vertices of P are themselves non-backtracking walks, P is a finite induced subgraph of the universal cover T, and A_P is a finite submatrix of A_T. By Theorem 2.4, the roots of μ_G are bounded by

$$\begin{aligned} \|A_P\|_2 &= \sup_{\|x\|_2=1} \|A_P x\|_2 \\ &\leq \sup_{\|y\|_2=1, \mathrm{supp}(y) \subset P} \|A_T y\|_2 \\ &\leq \sup_{\|y\|_2=1} \|A_T y\|_2 = \rho(T) \end{aligned}$$

as desired. □

One could directly prove an upper bound of $2\sqrt{d-1}$ on the spectral radius of a path tree of a d-regular graph and an upper bound of $\sqrt{c-1} + \sqrt{d-1}$ on the spectral radius of a path tree of a $(c,d)-$ regular bipartite graph without considering infinite trees. An identity of Godsil and Gutman [8] is that the expected characteristic polynomial of a random signing of the adjacency matrix of a graph is equal to its matching polynomial. To associate a signing of the edges of G with a vector in $\{\pm 1\}^m$, we choose an arbitrary ordering of the m edges of G, denote the edges by $e_1, ..., e_m$ and denote a signing of these edges by $s \in \{\pm 1\}^m$. We then let A_s denote the signed adjacency matrix corresponding to s, and define $f_s(x) = \det(xI - A_s)$ to be the characteristic polynomial of A_s.

Theorem 2.6. *[8]*

$$\mathbb{E}_{s \in \{\pm 1\}^m} [f_s(x)] = \mu_G(x)$$

To prove that a good lift exists, it suffices, by Theorems 2.2 and 2.6 , to show that there is a signing s so that the largest root of $f_s(x)$ is at most the largest root of $\mathbb{E}_{s \in \{\pm 1\}^m} [f_s(x)]$. To do this, we prove that the polynomials $\{f_s(x)\}_{s \in \{\pm 1\}^m}$ are what we call an interlacing family.

Lemma 2.7. *[1] Let A be an adjacency matrix of G, A_s be the signed adjacency matrix associated with a 2-lift $\hat{G}$. Then every eigenvalue of A and every eigenvalue of A_s are eigenvalues of $\hat{G}$. Furthermore, the multiplicity of each eigenvalue of $\hat{G}$ is the sum of multiplicities in A and A_s.*

2.2 Interlacing Families

Definition 2.8. *We say that a polynomial $g(x) = \prod_{i=1}^{n-1}(x - \alpha_i)$ interlaces a polynomial $f(x) = \prod_{i=1}^{n}(x - \beta_i)$ if*

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq ... \leq \alpha_{n-1} \leq \beta_n$$

We say that the polynomials $f_1, ..., f_k$ have a common interlacing if there is a polynomial g so that g interlaces f_i for each i.

Let $\beta_{i,j}$ be the j^{th} smallest root of f_i. The polynomials $f_1, ..., f_k$ have a common interlacing if and only if there are numbers $\alpha_0 \leq \alpha_1 \leq ... \leq \alpha_n$ so that $\beta_{i,j} \in [\alpha_{j-1}, \alpha_j]$ for all i and j. The numbers $\alpha_1, ..., \alpha_{n-1}$ come from the roots of the polynomial g, and α_0 (α_n) can be chosen to be any number that is smaller (larger) than all of the roots of all of the f_i.

Lemma 2.9. *Let $f_1, ..., f_k$ be polynomials of the same degree that are real-rooted and have positive leading coefficients. Define*

$$f_\emptyset = \sum_{i=1}^{k} f_i$$

If $f_1, ..., f_k$ have a common interlacing, then exists an i so that the largest root of f_i is at most the largest roots of $f_\emptyset$.

Proof. Let the polynomials be of degree n. Let g be a polynomial that interlaces all of the f_i, and let α_{n-1} be the largest root of g. As each f_i has a positive leading coefficient, it is positive for sufficiently large x. As each f_i has exactly one root that is at least α_{n-1}, each f_i is non-positive at α_{n-1}. So, $f_\emptyset$ is also non-positive at α_{n-1}, and eventually becomes positive. This tells us that $f_\emptyset$ has a root that is at least α_{n-1}, and so its largest root is at least α_{n-1}. Let β_n be this root.

As $f_\emptyset$ is the sum of the f_i, there must be some i for which $f_i(\beta_n) \geq 0$. As f_i has at most one root that is at least α_{n-1}, and $f_i(\alpha_{n-1}) \leq 0$, the largest root of f_i is it at least α_{n-1} and at most β_n. □

Note that if the polynomials do not have a common interlacing, the sum may fail to be real rooted.

Definition 2.10. *Let $S_1, ..., S_m$ be finite sets and for every assignment $s_1, ..., s_m \in S_1 \times ... \times S_m$, let $f_{s_1,...,s_m}(x)$ be a real-rooted degree n polynomial with positive leading coefficient. For a partial assignment $s_1, ..., s_k \in S_1 \times ... \times S_k$ with $k < m$, define*

$$f_{s_1,...,s_k} \stackrel{def}{=} \sum_{s_{k+1} \in S_{k+1}, ..., s_m \in S_m} f_{s_1,...,s_k,s_{k+1},...,s_m}$$

as well as

$$f_\emptyset \stackrel{def}{=} \sum_{s_1 \in S_1, ..., s_m \in S_m} f_{s_1,...,s_m}$$

We say that the polynomials $\{f_{s_1,...,s_m}\}_{s_1,...,s_m}$ *form an interlacing family if for all* $k = 0, ..., m-1$*, and all* $s_1, ..., s_k \in S_1 \times ... \times S_k$*, the polynomials*

$$\{f_{s_1,...,s_k,t}\}_{t \in S_{k+1}}$$

have a common interlacing.

Theorem 2.11. *Let* $S_1, ..., S_m$ *be finite sets and let* $\{f_{s_1,...,s_m}\}$ *be an interlacing family of polynomials. Then, there exists some* $s_1, ..., s_m \in S_1 \times ... \times S_m$ *so that the largest root of* $f_{s_1,...,s_m}$ *is less than the largest root of* $f_\emptyset$*.*

Proof. By definition of interlacing family, the polynomials $\{f_t\}$ for $t \in S_1$ have a common interlacing and their sum is $f_\emptyset$. By Lemma 2.9, one of the polynomials has largest root that is at most the largest root of $f_\emptyset$. Proceeding inductively, for any $s_1, ..., s_k$, we know that the polynomials $\{f_{s_1,...,s_k,t}\}$ for $t \in S_{k+1}$ have a common interlacing and that their sum is $f_{s_1,...,s_k}$. So, for some choice of t the largest root of the polynomial $f_{s_1,...,t}$ is at most the largest root of $f_{s_1,...,s_k}$. □

We will prove that the polynomials $\{f_s\}_{s \in \{\pm 1\}^m}$ as defined previously are an interlacing family. This requires establishing the existence of certain common interlacings as previously defined. We can do this using the fact that common interlacings are equivalent to real-rootedness statements.

We introduce the following lemma, which will be used in proving further results.

Lemma 2.12. *[4] Let* $f_1, ..., f_k$ *be (univariate) polynomials of the same degree with positive leading coefficients. Then* $f_1, ..., f_k$ *have a common interlacing if and only if* $\sum_{i=1}^k \lambda_i f_i$ *is real rooted for all convex combinations* $\lambda_i \geq 0, \sum_{i=1}^k \lambda_i = 1$.

The proof that the polynomials $\{f_s\}_{s \in \{\pm 1\}^m}$ form an interlacing family relies on the following generalization of the fact that the matching polynomial is real-rooted.

Theorem 2.13. *Let* $p_1, ..., p_m$ *be numbers in* $[0, 1]$*. Then, the following polynomial is real-rooted*

$$\sum_{s \in \{\pm 1\}^m} \left(\prod_{i: s_i = 1} p_i \right) \left(\prod_{i: s_i = -1} (1 - p_i) \right) f_s(x)$$

The proof for this theorem requires the use of real stable polynomials, which we will introduce later. This theorem immediately leads to the following result.

Theorem 2.14. *The polynomials* $\{f_s\}_{s \in \{\pm 1\}^m}$ *are an interlacing family.*

Proof. We first show that for every $0 \leq k \leq m-1$, every partial assignment $s_1 \in \pm 1, ..., s_k \in \pm 1$, and every $\lambda \in [0,1]$, the polynomial

$$\lambda f_{s_1,...,s_k,1}(x) + (1-\lambda) f_{s_1,...,s_k,-1}(x)$$

is real-rooted. To show this, we apply Theorem 2.13 with $p_{k+1} = \lambda, p_{k+2}, ..., p_m = 1/2$ and $p_i = (1+s_i)/2)$ for $1 \leq i \leq k$. We then apply Lemma 2.12. □

Theorem 2.15. *Let G be a graph with adjacency matrix A and universal cover T. Then there is a signing of s such that all the eigenvalues of A_s are at most $\rho(T)$. In particular, if G is $d-$regular, there is a signing s so that the eigenvalues of A_s are at most $2\sqrt{d-1}$.*

Proof. The first statement follows immediately from Theorems 2.11 and 2.14 and Lemma 2.5 , The second statement follows by noting that the universal cover of a $d-$regular graph is the infinite $d-$regular tree, which has spectral radius at most $2\sqrt{d-1}$. □

Lemma 2.16. *Every non-trivial eigenvalue of a complete $(c,d)-$biregular graph is zero.*

Proof. The adjacency matrix A of this graph has rank 2. Hence all of its eigenvalues other than $\pm\sqrt{cd}$ is zero. □

Theorem 2.17. *For ever $d \geq 3$, there is an infinite sequence of $d-$regular bipartite Ramanujan graphs.*

Proof. We know from Lemma 2.16 that the complete bipartite graph of degree d is Ramanujan. Using Lemma 2.7 and Theorem 2.15, for every $d-$regular graph bipartite Ramanujan graph G, there is a 2-lift in which every non-trivial eigenvalue is at most $2\sqrt{d-1}$. As the 2-lift of a bipartite graph is bipartite, and the eigenvalues of a bipartite graph are symmetric about 0, this 2-life is also a regular bipartite Ramanujan graph.

Hence, for every $d-$regular bipartite Ramanujan graph G, there is another $d-$regular bipartite Ramanujan graph with twice as many vertices. □

2.3 Real stable polynomials and proof of Theorem 2.13

Definition 2.18. *A multivariate polynomial $f \in \mathbb{R}[z_1, ..., z_n]$ is called real stable if it is the zero polynomial or if*

$$f(z_1, ..., z_n) \neq 0$$

whenever the imaginary part of every z_1 is strictly positive.

A real stable polynomial has real coefficients, but may be evaluated on complex input.

Lemma 2.19. *[2] Let $A_1, ..., A_m$ be positive semidefinite matrices. Then*

$$\det(z_1 A_1 + ... + z_m A_m)$$

is real stable.

There are some nice properties for real stable polynomials. In particular, if $f(x_1, ..., x_k)$ and $g(y_1, ..., y_j)$ are real stable, then $f(x_1, ..., x_k)g(y_1, ..., y_j)$ is real stable.

In this section, for a vriable x_i, we let Z_{x_i} be the operator on polynomials induced by setting this variable to zero. We will also let ∂_{z_i} be the operation of partial differentiation with respect to z_i. For $\alpha, \beta \in \mathbb{N}^n$, we use the notation

$$z^\alpha = \prod_{i=1}^{n} z_i^{\alpha_i} \quad \text{and} \quad \partial^\beta = \prod_{i=1}^{n} (\partial_{z_i})^{\beta_i}.$$

Theorem 2.20. *[3] Let $T : \mathbb{R}[z_1, ..., z_n] \to \mathbb{R}[z_1, ..., z_n]$ be an operator of the form*

$$T = \sum_{\alpha,\beta \in \mathbb{N}^n} c_{\alpha,\beta} z^\alpha \partial^\beta$$

where $c_{\alpha,\beta} \in \mathbb{R}$ and $c_{\alpha,\beta}$ is zero for all but finitely many terms. Define

$$F_T(z, w) := \sum_{\alpha,\beta} c_{\alpha,\beta} z^\alpha w^\beta$$

Then T presrves real stability if and only if $F_T(z, -w)$ is real stable.

A special case of this result is the following corollary

Corollary 2.21. *For non-negative real numbers p, q and variables u, v, the operator $T = 1 + p\partial_u + q\partial_v$ preserves real stability.*

Proof. To show that the polynomial $1 - pu - qv$ is real stable, consider u, v with positive imaginary parts. The imaginary part of $1 - pu - qv$ will then be negative, and cannot be zero. □

Now, we will show how operators of the preceding kind can be used to generate the expected characteristic polynomials that appears in Theorem 2.13.

Lemma 2.22. *For an invertible matrix A, vectors a and b, and $p \in [0, 1]$,*

$$Z_u Z_v (1 + p\partial_u + (1-p)\partial_v) \det(A + uaa^T + vbb^T) = p \det(A + aa^T) + (1-p) \det(A + bb^T).$$

Proof. Note that from matrix determinant lemma, for every nonsingular matrix A and every real number t,

$$\det(A + taa^T) = \det(A)(1 + ta^T A^{-1} a).$$

One consequence of this is Jacobi's formula for the derivative of the determinant:

$$\partial_t \det(A + taa^T) = \det(A)(a^t A^{-1} a).$$

This implies

$$\begin{aligned} &Z_u Z_v (1 + p\partial_u + (1-p)\partial_v) \det(A + uaa^T + vbb^T) \\ &= \det(A)(1 + p(a^T A^{-1} a) + (1-p)(b^T A^{-1} b)) \end{aligned}$$

By matrix determinant lemma, this equals to

$$p \det(A + aa^T) + (1-p)\det(A + bb^T).$$

□

This allows us to prove the following theorem, which is used to prove Theorem 2.13.

Theorem 2.23. *Let $a_1, ..., a_m$ and $b_1, ..., b_m$ be vectors in $\mathbb{R}^n$, and let $p_1, ..., p_m$ be real numbers in $[0,1]$, and let D be a positive semidefinite matrix. Then every univariate polynomial of the form*

$$P(x) \overset{def}{=} \sum_{S \subseteq [m]} \left(\prod_{i \in S} p_i \right) \left(\prod_{i \notin S} 1 - p_i \right) \det \left(xI + D + \sum_{i \in S} a_i a_i^T + \sum_{i \notin S} b_i b_i^T \right)$$

is real-rooted.

Proof sketch. Let $u_1, ..., u_m$ and $v_1, ..., v_m$ be variables and define

$$Q(x, u_1, ..., u_m, v_1, ..., v_m) = \det \left(xI + D + \sum_i u_i a_i a_i^T + \sum_i v_i b_i b_i^T \right)$$

By Lemma 2.19, Q is real stable.

We then prove by induction that P can be rewritten as

$$P(x) = \left(\prod_{i=1}^m Z_{u_i} Z_{v_i} T_i \right) Q\left(x, u_1, ..., u_m, v_1, ..., v_m\right)$$

where $T_i = 1 + p_i \partial_{u_i} + (1-p_i)\partial_{v_i}$. The inductive step follows from Lemma 2.22. We can then apply Corollary 2.21 and closure of real stable polynomials under the restrictions of variables to real constants to see that each of the polynomials above is real stable. Since $P(x)$ is real stable and has one variable, it is real-rooted. □

With these tools, we are able to prove Theorem 2.13.

Proof of Theorem 2.13. For each vertex u, let d_u be its degree, and let $d = \max_u d_u$. We need to prove that the polynomial

$$\sum_{s\in\{\pm 1\}^m} \left(\prod_{i:s_i=1} p_i\right)\left(\prod_{i:s_i=-1} (1-p_i)\right) \det(xI - A_s)$$

is real-rooted. This is equivalent to proving that the the following polynomial is real-rooted

$$\sum_{s\in\{\pm 1\}^m} \left(\prod_{i:s_i=1} p_i\right)\left(\prod_{i:s_i=-1} (1-p_i)\right) \det(xI + dI - A_s)$$

as their roots only differ by d. We now observe that the matrix $dI - A_s$ is a signed Laplacian matrix of G plus a nonnegative diagonal matrix. For each edge (u, v), define the rank 1-matrices as follows:

$$L^1_{u,v} = (e_u - e_v)(e_u - e_v)^T,$$
$$L^{-1}_{u,v} = (e_u + e_v)(e_u + e_v)^T$$

where e_u is the elementary unit vector in direction u. Consider a signing s and let $s_{u,v}$ denote the sign it assigns to edge (u, v). Since the original graph had maximum degree d, we have

$$dI - A_s = \sum_{(u,v)\in E} L^{s_{u,v}}_{u,v} + D$$

where D is the diagonal matrix whose u^{th} diagonal entry equals $d - d_u$. As the diagonal entries of D are non-negative, it is positive semidefinite. If we now set $a_{u,v} = (e_u - e_v)$ and $b_{u,v} = (e_u + e_v)$, we can then express the polynomial as follows:

$$\sum_{s\in\{\pm 1\}^m} \left(\prod_{i:s_i=1} p_i\right)\left(\prod_{i:s_i=-1} (1-p_i)\right) \det(xI + dI - A_s)$$
$$= \sum_{s\in\{\pm 1\}^m} \left(\prod_{i:s_i=1} p_i\right)\left(\prod_{i:s_i=-1} (1-p_i)\right) \det\left(xI + D + \sum_{s_{u,v}=1} a_{u,v}a^T_{u,v} + \sum_{s_{u,v}=-1} b_{u,v}b^T_{u,v}\right)$$

It then follows from Theorem 2.23 that this polynomial is real-rooted. □

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