Lecture Series 3

October 5, 2024

In this chapter, we will use $\mathcal{H} = \{z : \text{Im}(z) > 0\}$ to denote the open upper half plane, and $\boldsymbol{z}$ to be the vector $\{z_1, ..., z_n\}$.

1 Stable Polynomials

We have previously defined the notion of real stable polynomials and introduced some Lemmas and Theorems that was used to prove Theorem 2.13 in Chapter 2.

Definition 1.1. *A polynomial $f(\boldsymbol{z})$ is real stable (resp. stable) if and only if for every $\mathbf{e} \in \mathbb{R}^n_{>0}$ and $\boldsymbol{x} \in \mathbb{R}^n$, the univariate restriction*

$$t \mapsto f(t\mathbf{e} + \boldsymbol{x})$$

is real-rooted (resp. stable).

Lemma 1.2. *For positive semidefinite matrices $A_1, ..., A_n \succeq 0$ and Hermitian B, the determinantal polynomial*

$$\det\left(\sum_{i=1}^{n} z_i A_i + B\right)$$

is real stable.

Proof sketch. Assume that A_i are positive definite and consider a univariate restriction

$$t \mapsto \det\left(t\sum_{i=1}^{n} e_i A_i + \left(\sum_{i=1}^{n} x_i A_i + B\right)\right)$$

Since the e_i are positive, $M := \sum_{i=1}^{n} e_i A_i \succ 0$ has a negative aquare root $M^{-1/2}$ and we may write the above as

$$t \mapsto \det\left(M^{-1/2}\right)\det\left(tI + M^{1/2}\left(\sum_{i=1}^{n} x_i A_i + B\right)M^{1/2}\right)\det\left(M^{-1/2}\right)$$

Since this is a multiple of a characteristic polynomial of a Hermitian matrix, it must be real-rooted.

The positive semidefinite case could be handled by taking a limit of positive definite matrices. Recall that the limit along each univariate restriction must be real-rooted or zero. □

Theorem 1.3. *[2] If $q(x,y)$ is a real stable polynomial of degree d, then there are real symmetric $d \times d$ positive semidefinite matrices A, B and symmetric matrix C such that*

$$q(x,y) = \pm \det(xA + yB + C).$$

This theorem is known to be false for more than 2 variables.

Example 1.4. *Let $G = (V, E)$ be a connected undirected graph. Then the spanning tree polynomial*

$$P_G(\boldsymbol{z}) = \sum_{\textit{spanning tree } T \subset E} \prod_{e \in T} z_e$$

is real stable.

The following Theorem states the closure properties of some linear transformations.

Theorem 1.5. *The following linear transformations on $\mathbb{C}[\boldsymbol{z}]$ maps every stable polynomial to another stable polynomial or to zero.*

1. *Permutation.* $f(z_1, z_2, ..., z_n) \mapsto f(z_{\sigma(1)}, ..., z_{\sigma(n)})$ *for some permutation* $\sigma : [n] \to [n]$.
2. *Scaling.* $f(z_1, ..., z_n) \mapsto f(az_1, ..., z_n)$ *where* $a > 0$.
3. *Diagonalization.* $f(z_1, z_2, ..., z_n) \mapsto f(z_2, z_2, z_3, ..., z_n) \in \mathbb{C}[z_2, \ldots, z_n]$.
4. *Inversion.* $f(z_1, z_n) \mapsto z_1^d f(-1/z_1, \ldots, z_n)$ *where* $d = \deg_1(f)$ *is the degree of* z_1 *in* f.
5. *Specialization.* $f \mapsto f(a, z_2, ..., z_n) \in \mathbb{C}[z_2, ..., z_n]$ *where* $a \in \mathcal{H} \cup \mathbb{R}$.
6. *Differentiation.* $f \mapsto \frac{\partial}{\partial z_1} f$.

Proof. (1) to (3) follows from definition.
(4) follows because $z \mapsto -1/z$ preserves the upper half plane.
(6) is a consequence of the Gauss-Lucas Theorem below. □

Theorem 1.6 (Gauss-Lucas)**.** *If $f \in \mathbb{C}[z]$, the roots of $f'(z)$ lie in the convex hull of the roots of $f(z)$.*

Proof. Let $\lambda_1, ..., \lambda_n \in \mathbb{C}$ be the roots of f and assume WLOG that f and f' have no common roots. If $f' = 0$, then we have

$$0 = \frac{f'(z)}{f(z)} = \sum_{i=1}^{n} \frac{1}{z - \lambda_I} = \sum_{i=1}^{n} \frac{\overline{z - \lambda_i}}{|z - \lambda_i|^2}$$

Rearranging, we obtain

$$z = \sum_{i=1}^{n} \frac{|z - \lambda_i|^{-2}}{\sum_{i=1}^{n} |z - \lambda_i|^{-2}} \lambda_i$$

which is a convex combination. □

1.1 A probabilistic application

1.1.1 Poisson Binomial Distribution

The distribution of a sum of independent Bernoulli random variables is called a Poisson Binomial Distribution, that is

$$X = \sum_{i=1}^{n} X_i$$

where X_i are independent Bernoullis with $\mathbb{E}(X_i) = b_i \in (0, 1)$, and taking $p_k = \mathbb{P}[X = k]$. We are interested in knowing if such a distrubtion is unimodal, that is whether there is some m such that $p_0 \leq p_1 \leq ... \leq p_m \geq ... \geq p_n$.

Consider the following generating function of the distribution

$$q(x) \stackrel{\text{def}}{=} \sum_{k=0}^{n} p_k x^k = \prod_{i=1}^{n} (b_i x + (1 - b_i))$$

where the independence of X_i yields a factorization of q(x) into linear terms. This factorization implies that $q(x)$ is real-rooted with strictly negative roots $\lambda_i := -\frac{1-b_i}{b_i} < 0$. Using the following Newton's Inequalities, which states

Theorem 1.7 (Newton Inequalities)**.** *If $\sum_{k=0}^{n} a_k x^k$ is real-rooted, then*

$$\left(\frac{a_k}{\binom{n}{k}} \right)^2 \geq \frac{a_{k-1}}{\binom{n}{k-1}} \frac{a_{k+1}}{\binom{n}{k+1}}$$

for $k = 1, ..., n - 1$.

After cancellation of the factorials, it reduces to

$$a_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) a_{k-1} a_{k+1},$$

which is strictly stronger than

$$a_k^2 \geq a_{k_1} a_{k+1} \qquad \text{(log-concavity)}$$

This implies unimodality, where the probabilities p_k must be unimodal. Now we introduce the next proposition, which sill be used in proving the next Theorem.

Proposition 1.8. *Suppose $p(x) = \sum_{k=0}^{n} a_k x^k$ is a real-rooted polynomial with nonnegative coefficients and $a_0 \neq 0), p(1) = 1$. Then there are independent Bernoulli random variables $X_1, ..., X_n$ such that*

$$a_k = \mathbb{P}\left[\sum_{i=1}^{n} X_i = k\right]$$

Proof. Factor $p(x)$ as $C\prod_{i=1}^{n}(x+\lambda_i)$ for some $\lambda_i > 0$. Since $p(1) = 1$, we must have

$$C = \frac{1}{\prod_{i=1}^{n}(1+\lambda_i)}$$

Then we have

$$p(x) = \prod_{i=1}^{n}(b_i x + (1 - b_i))$$

for $b_i = \frac{1}{1+\lambda_i} \in (0, 1)$. Taking X_i with $\mathbb{E}(X_i) = b_i$ proves the claim. □

1.1.2 Application

Suppose $G = (V, E)$ is a graph, $F \subset E$ is a cut, and T is a uniformly random spanning tree of G. The distribution of the random variable $|F \cap T|$ is a Poisson Binomial Distribution.

Theorem 1.9. *The distribution of $|F \cap T|$ is a Poisson Binomial Distribution.*

Proof. The generating polynomial of a random variable $T \cap F$ is obtained by setting all the variables $\mathbf{z}_e, e \notin 1$:

$$Q_G(\mathbf{z}|_F) = P_G(\mathbf{z}_F, 1, ..., 1),$$

where we observe that the coefficient of the monomial $z^S := \prod_{e \in S} z_e$ in Q_G is equal to the number of spanning trees T for which $T \cap F = S$. As setting the variables to real numbers preserve stability, Q_G is real stable. Thus its diagonal restriction

$$Q_G(x, x, ..., x) = \sum_{k=0}^{|F|} x^k \mathbb{P}\left[|T \cap F| = k\right]$$

must be real-rooted. Normalizing by $Q_G(1, 1, ..., 1)$ and applying the previous proposition finishes the proof. □

1.2 Characterization of Stability Preserving Operators

Let $\mathbb{C}_k[z_1, ..., z_n]$ be the vector space of complex polynomials in $z_1, ..., z_n$ in which each variable has degree of at most k. We call a linear transformation nondegenerate if its range has dimension at least 2.

Theorem 1.10. *A nondegenerate linear operator $T : \mathbb{C}_k [z_1, ..., z_n] \to \mathbb{C} [z_1, ..., z_n]$ preserves stability iff the $2n$-variate polynomial*

$$G_T (z_1, ..., z_n, w_1, ..., w_n) := T \left[(z_1 + w_1)^k ... (z_n + w_n)^k\right]$$

is stable, where the operator T only acts on the z variables.

This theorem says that there is a single $2n-$variate polynomial whose stability guarantees the stability of all of the $n-$variate images $T(f)$.

1.2.1 Heilmann-Lieb Theorem

Given a graph with nonnegative edge weights $w_e \geq 0, e \in E$, we define

$$m_k := \sum_{\text{matching } M, |M|=k} \prod_{e \in M} w_e$$

The relevant generating function is the matching polynomial

$$\mu_G(x) := \sum_{k=0}^{n/2} x^{n-2k} (-1)^k m_k.$$

We state the following theorem:

Theorem 1.11 (Heilmann-Lieb)**.** *For every weighted graph G with nonnegative edge weights, $\mu_G(x)$ is real-rooted.*

Proof. Given a graph G with positive edge weights $w_{uv} > 0, uv \in E$, consider the multivariate polynomial

$$Q_G(\mathbf{z}) = \prod_{uv \in E} (1 - w_{uv} z_u z_v),$$

where the variables z_v are indexed by $v \in V$. As Q_G is a product of real stable polynomials, it is real stable. Consider the multiaffine part operator

$$\text{MAP} : \mathbb{C} [\mathbf{z}] \to \mathbb{C}_1 [\mathbf{z}]$$

defined on monomials of degree at most $m := |E|$ in each variable by

$$\text{MAP} \left(\prod_{e \in S} z_e^{d_e} \right) = \begin{cases} \prod_{e \in S} z_e & \text{if } d_e \leq 1 \text{ for all } e \\ 0 & \text{otherwise} \end{cases}$$

The symbol of this operator is given by

$$G_{\mathrm{MAP}}(\mathbf{z},\mathbf{w}) = \mathrm{MAP}\left(\prod_{v\in V}(z_v+w_v)^m\right) = \prod_{v\in V}\left(w_v^m + m z_v w_v^{m-1}\right) = \prod_{v\in V} w_v^{m-1}\left(w_v + m z_v\right),$$

which is real stable. Since MAP is nondegenerate, Theorem 1.10 states that it preserves stability. Thus,

$$\mathrm{MAP}\left(Q_G\right) = \sum_{\text{matching } M} (-1)^{|M|} \prod_{\text{edge } uv\in M} w_{uv} \prod_{\text{vertex } v\in M} z_v$$

is real stable, and its univariate diagonal restriction $z_v \leftarrow x, v \in V$:

$$\sum_{\text{matching } M} (-1)^{|M|} x^{2|M|} \prod_{uv\in M} w_{uv}$$

is real-rooted. But this is just the reversal of $\mu_G(x)$ $\square$

2 Multiaffine Real Stable Polynomials

A multiaffine polynomial is a multivariate polynomial in which each variable has degree at most one. We use $\mathbb{R}_1[z_1,...,z_n]$ or $\mathbb{R}_{MA}[z_1,...,z_n]$ to denote vector spaces of multiaffine polynomials.

Definition 2.1. *$f \in \mathbb{R}[z_1,...,z_n]$ is Strongly Rayleigh if for every $i \neq j$,*

$$\partial_{z_i} f(\boldsymbol{x}) \cdot \partial_{z_j} f(\boldsymbol{x}) \geq \partial_{z_i z_j} f(\boldsymbol{x}) \cdot f(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \mathbb{R}^n.$$

Theorem 2.2. *[1] A real multiaffine polynomial $f \in \mathbb{R}_1[z_1,...,z_n]$ is stable if and only if it is Strongly Rayleigh.*

Proof. ($\Rightarrow$) Suppose f is real stable. Fix $\boldsymbol{x} \in \mathbb{R}^n$. Consider the bivariate restriction

$$g(s,t) := f(\boldsymbol{x} + s e_i + t e_j)$$

which is a multiaffine bivariate polynomial

$$g(s,t) = a + bs + ct + dst$$

with real coefficients

$$a = f(\boldsymbol{x}) \quad b = \partial_{z_i} f(\boldsymbol{x}) \quad c = \partial_{z_j} f(\boldsymbol{x}) \quad d = \partial_{z_i z_j} f(\boldsymbol{x})$$

Since every univariate restriction of g along a direction in the positive orthant $\mathbb{R}^2_{>0}$ is a restriction of a specialization of f (by fixing all the variables other than z_i, z_j), all such restrictions are real-rooted and g is itself real stable. Observe by applying the closure properties that for every $\lambda > 0$, the polynomial $g(\lambda r, r) = a + (\lambda b + c) r + d\lambda r^2$ must be real-rooted, hence $(\lambda b + c)^2 \geq 4ad\lambda$ for all $\lambda > 0$. If b and c are nonzero of the same sign then setting $\lambda = c/b$ yields the inequality. If they have opposite signs or if one of them is zero then we can see that g cannot be real stable unless it is zero.

($\Leftarrow$) We prove this by induction. Suppose $f(\boldsymbol{z}, z_{n+1}) = g(\boldsymbol{z}) + z_{n+1}h(\boldsymbol{z})$ is Strongly Rayleigh, with $g, h \in \mathbb{R}_1[z_1, ..., z_n]$. Note that both $g(\boldsymbol{z}), h(\boldsymbol{z})$ are Strongly Rayleigh by closure properties. Let $z_{n+1} = \alpha \in \mathbb{R}$. Observe that $g(\boldsymbol{z}) + \alpha h(\boldsymbol{z}) \in \mathbb{R}_1[\boldsymbol{z}]$ is Strongly Rayleigh. By induction, it must be stable for every α. If it is identically zero for some α, then $g(\boldsymbol{z}) \equiv -\alpha h(\boldsymbol{z})$ and we may factor f as $f(\boldsymbol{z}, z_{n+1} = (z_{n+1} - \alpha)h(\boldsymbol{z})$, which is stable and we are done.

Otherwise, $g(\boldsymbol{z}) + \alpha h(\boldsymbol{z}) \neq 0$ for all α and for all $\boldsymbol{z} \in \mathcal{H}^n$. This means that

$$\Phi(\boldsymbol{z}) := \frac{g(\boldsymbol{z})}{h(\boldsymbol{z})} \notin \mathbb{R} \quad \forall \boldsymbol{z} \in \mathcal{H}^n$$

Since Φ is continuous on $\mathcal{H}^n$ we must have either

$$\begin{aligned} \operatorname{Im}(\Phi(\boldsymbol{z})) > 0 \quad \forall \boldsymbol{z} \in \mathcal{H}^n \quad \text{or} \\ \operatorname{Im}(\Phi(\boldsymbol{z})) < 0 \quad \forall \boldsymbol{z} \in \mathcal{H}^n \end{aligned}$$

In the latter case it is immediate that $f(\boldsymbol{z}, z_{n+1}$ is stable. In the former, we find that by changing the sign of z_{n+1}, $f(\boldsymbol{z}, -z_{n+1})$ must be stable. By the forward direction of the theorem proven earlier, this means that it is Strongly Rayleigh. Applying the definition of Strongly Rayleigh to the pairs $i, n+1$, we obtain the reversed inequalities:

$$\partial_{z_i} f(\boldsymbol{x}) \cdot \partial_{z_{n+1}} f(\boldsymbol{x}) \leq \partial_{z_i z_{n+1}} f(\boldsymbol{x}) \cdot f(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \mathbb{R}^n$$

Since f is also Strongly Rayleigh these must be equalities. We could check that this is only possible when $g = h$. □

2.1 Multiaffine Stability Preservers

In this subsection, we derive a sufficient condition for establishing that a linear transformation on $\mathbb{C}_1[z_1, ..., z_n]$ preserves stability. The main purpose of this subsection is to show that a transformation T preserves stability of $n-$variate polynomials. It suffice to show that the stability of a single $2n-$variate generating polynomial derived from it.

Lemma 2.3 (Lieb-Sokal). *Suppose $f(\boldsymbol{z}) + wg(\boldsymbol{z}) \in \mathbb{C}[\boldsymbol{z}, w]$ is stable and the degree of z_1 in g is at most 1. Then*

$$f(\boldsymbol{z}) - \partial_{z_1} g(\boldsymbol{z}) \in \mathbb{C}[\boldsymbol{z}]$$

is stable.

Proof. By closure properties, f and g must be stable. Let $\alpha \in \mathcal{H}$. Observe that $-\alpha^{-1} \in \mathcal{H}$. Hence

$$h(\boldsymbol{z}, \alpha) := \alpha g(z_1 - \alpha^{-1}, z_2, ..., z_n) \neq 0$$

for all $\boldsymbol{z} \in \mathcal{H}^n$. Since g is affine in z_1, then h is a multiaffine stable polynomial. Expanding the first variable, we find that

$$h(\boldsymbol{z}, \alpha) = \alpha g(\boldsymbol{z}) - (\partial_{z_1} g)(\boldsymbol{z})$$

is stable. Rearranging, we have

$$\operatorname{Im}\left(\frac{-\partial_{z_1} g(\boldsymbol{z})}{g(\boldsymbol{z})}\right) \geq 0$$

for all $\boldsymbol{z} \in \mathcal{H}^n$. Similarly, since $f + wg$ is stable, we know that $\operatorname{Im}(f(\boldsymbol{z})/g(\boldsymbol{z})) \geq 0$ for all $\boldsymbol{z} \in \mathcal{H}^n$. Hence, summing up we ahve

$$\operatorname{Im}\left(\frac{-\partial_{z_1} g(\boldsymbol{z}) + f(\boldsymbol{z})}{g(\boldsymbol{z})}\right) \geq 0 \quad \forall \boldsymbol{z} \in \mathcal{H}^n,$$

so $\partial_{z_1} g(\boldsymbol{z}) + f(\boldsymbol{z}) + vg(\boldsymbol{z}) \in \mathbb{R}[\boldsymbol{z}, v]$ is stable. Specializing v to 0 gives the result. $\square$

This Lemma is used in the proof of the following Borcea-Branden Theorem, which follows because every linear operator on a space of bounded degree polynomials can be written as an appropriate sum of differential operators.

Theorem 2.4 (Borcea-Branden)**.** *Suppose $T : \mathbb{C}_1[\boldsymbol{z}] \to \mathbb{C}[\boldsymbol{z}]$ is a linear operator. If the algebraic symbol*

$$G_T(z_1, ..., z_n, w_1, ..., w_n) := T\left(\prod_{j=1}^{n}(z_j + w_j)\right) = \sum_{S \subset [n]} T(z^S) w^{[n] \setminus S}$$

is stable then T is stability-preserving.

Proof. Suppose $f(\boldsymbol{z}) = \sum_{S \subset [n]} a_S z^S \in \mathbb{C}_1[\boldsymbol{z}]$ is stable. Since $w_j \mapsto -1/w_j$ preserves $\mathcal{H}$, the hypothesis implies that

$$w_1 ... w_n G(z_1, ..., z_n, -1/w_1, ..., -1/w_n) = \sum_{S \subset [n]} T(z^S)(-1)^{n-|S|} w^S$$

is stable. Multiplying by $f(v_1, ..., v_n) \in \mathbb{C}_1[v_1, ...,]$, we find that

$$\sum_{S \subset [n]} T(z^S)(-1)^{n-|S|} w^S f(v_1, ..., v_n)$$

is stable. Since f is multiaffine, we can use Lieb-Sokal Lemma to replace each w_i by $-\partial_{v_i}$, which shows

$$\sum_{S \subset [n]} T(z^S)(-1)^{n-|S|}(-1)^{|S|}(\partial_v^S) f(v_1, ..., v_n)$$

is stable, where $\partial_v^S := \prod_{j\in S}\partial_{v_j}$. Setting $v_1 = ... = v_n = 0$ preserves stability. So

$$\sum_{S\subset[n]} T(z^S)(-1)^n(\partial_v^S)f(0,...,0)$$

is stable. Since $(\partial_v^S f = a_S$, the expression is equal to

$$(-1)^n \sum_{S\subset[n]} T(z^S)a_S = (-1)^n T(f).$$

Thus T preserves stability as we desire. □

2.2 Polarization and High Degree Polynomials

Polarization is a tool that allows one to transfer results about multiaffine polynomials to polynomials of higher degree.

Definition 2.5. *Given a polynomial $f \in \mathbb{C}_k[z_1,...,z_n]$, the polarization of f is the unique polynomial*

$$F \in \mathbb{C}_1[z_{11},...,z_{1k},...,z_{n1},...,z_{nk}]$$

such that

1. *The restriction $z_{ji} \leftarrow z_j, j = 1,...,n$ is equal to the original polynomial:*

$$F(z_1,...,z_1,...,z_n,...,z_n) = f(z_1,...,z_n)$$

2. *For every $j = 1,...,n$, F is symmetric in $z_{j1},...,z_{jn}$.*

The polarization operation is denoted by $F = \Pi_k^{\uparrow}(f)$. The inverse is called projection and denoted by $f = \Pi_k^{\downarrow}(F)$. It is trivial to see that if F is stable then $f = \Pi_k^{\downarrow}(F)$ is also stable. The converse is also true, which is the following theorem.

Theorem 2.6. *If $f \in \mathbb{C}_k[z_1,...,z_n]$ is stable then $\Pi_k^{\uparrow}(f)$ is also stable.*

The proof is a result of the following Lemma:

Lemma 2.7. *If $f \in \mathbb{C}_1[z_1,...,z_n]$ is stable, then for every $\theta \in [0,1]$:*

$$(1-\theta)f(z_1,z_2,...,z_n) + \theta f(z_2,z_1,...,z_n)$$

is stable.

Proof. Setting all the variables other than z_1, z_2 to values in $\mathcal{H}$, it is sufficient to prove the claim for bivariate polynomials. We notice that

$$T : g(z_1,z_2) \mapsto (1-\theta)g(z_1,z_2) + \theta g(z_2,z_1)$$

is a linear operator on $\mathbb{C}_1 [z_1, z_2]$. Its symbol is

$$\begin{aligned} G_T(z_1, z_2, w_1, w_2) &= T((z_1 + w_1)(z_2 + w_2)) \\ &= z_1 z_2 + w_1((1-\theta)z_2 + \theta z_1) + w_2((1-\theta)z_1 + \theta z_2) + w_1 w_2 \end{aligned}$$

We will show that G_T is Strongly Rayleigh, which also implies that it is stable, which finishes the proof by Theorem 2.4. By symmetry, we only need to check the following two inequalities:

$$\partial_{z_1} G_T \cdot \partial_{z_2} G_T - \partial_{z_1 z_2} G_T \cdot G_T \geq 0$$

and

$$\partial_{z_1} G_T \cdot \partial_{w_1} G_T - \partial_{z_1 w_1} G_T \cdot G_T \geq 0$$

where the polynomials are evaluated at real points. We could simplify the expressions by computation to $\theta(1-\theta)(w_1 - w_2)^2$ and $\theta(z_1 - w_1)^2$ respectively. Hence they must be nonnegative. □

Proof sketch of Theorem 2.6. Let $T_{ij,\theta}$ be the partial symmetrization operator which swaps indices i and j with probability θ. It can be shown by induction that for every n there is a finite sequence of pairs $i_1 j_1, ..., i_N j_N$ and numbers $\theta_1, ..., \theta_N$ so that for every polynomial $f(z_1, ..., z_n)$:

$$T_{i_N j_N, \theta_N} ... T_{i_1 j_1, \theta_1} f = \mathbb{E}_\sigma \left(f \left(z_{\sigma(1)}, ..., z_{\sigma(n)} \right) \right) =: \mathrm{Sym}(f)$$

where the expectation is taken over a random permutation σ of $[n]$ - i.e., we can generate a uniformly random permutation by performing appropriately biased swaps on a predetermined sequence of pairs. Hence the symmetrization operator $\mathrm{Sym}(f)$ preserves stability.

Let $\Pi^{\uparrow}_{k,j}$ where $j = 1, \ldots, n$ be the operator which polarizes the variable z_j only, and note that $\Pi^{\uparrow}_k = \Pi^{\uparrow}_{k,n} \circ \ldots \circ \Pi^{\uparrow}_{k,1}$. Thus it is sufficient to show that $\Pi^{\uparrow}_{k,1}$ preserves stability. By setting every $z_j, j \neq 1$ to a number in $\mathcal{H}$, it suffices to handle the univariate case. Thus we let

$$g(z) = C \prod_{i=1}^{k} (z - \alpha_i)$$

be a univariate stable polynomial. As each $\alpha_i \notin \mathcal{H}$, each the polynomials $z_i - \alpha_i$ are stable, hence the product

$$G(z_1, ..., z_k) = C \prod_{i=1}^{k} (z_i - \alpha_i)$$

is stable. By the previous paragraph, $\mathrm{Sym}(G)$ must be stable; but as $\mathrm{Sym}(G)$ is symmetric in $z_1, ..., z_k$ and projects to g, it is equal to the polarization of g. □

Hence a polynomial is stable iff its polarization is stable.

Theorem 2.8. *Suppose $T : \mathbb{C}_k [z_1, \ldots, z_n] \to \mathbb{C} [z_1, \ldots, z_n]$ is a linear transformation and*

$$G_T(z_1, ..., z_n, w_1, ..., w_n) := T \left[(z_1 + w_1)^k ... (z_n + w_n)^k \right]$$

is stable. Then T preserves stability.

Proof. Define an operator $\Pi_k^{\uparrow}(T) : \mathbb{C}_1 [z_{11}, ..., z_{nk}] \to \mathbb{C}_1 [z_{11}, ..., z_{nk}]$ by

$$\Pi_k^{\uparrow}(T)(f) = \Pi_k^{\uparrow} \circ T \circ \Pi_k^{\downarrow}(f)$$

It is simple to check that $G_{\Pi_k^{\uparrow}}(T) = \Pi_k^{\uparrow}(G_T)$. Since G_T is stable, Theorem 2.6 implies that $G_{\Pi_k^{\uparrow}}(T) = \Pi_k^{\uparrow}(G_T)$ is stable. Hence by Theorem 2.4, $\Pi_k^{\uparrow}(T)$ presrves stability. But $T = \Pi_k^{\downarrow} \circ \Pi_k^{\uparrow}(T) \circ \Pi_k^{\uparrow}$ which by Theorem 2.6 preserves stability. $\square$

References

[1] Petter Brändén. Polynomials with the half-plane property and matroid theory. *Advances in Mathematics*, 216(1):302–320, 2007.

[2] Adrian Lewis, Pablo Parrilo, and Motakuri Ramana. The lax conjecture is true. *Proceedings of the American Mathematical Society*, 133(9):2495–2499, 2005.