Lecture Series 5

August 29, 2025

Here, we will discuss on hyperbolicity and a related conjecture.

1 Homogeneous Polynomials

Definition 1.1. *A homogeneous complex polynomial in n variables of degree d is called hyperbolic in direction $e \in \mathbb{R}^n$ if $p(e+iy) \neq 0 \quad \forall y \in \mathbb{R}^d$. Equivalently, if*

1. $p(e) \neq 0$
2. *The univariate polynomial $p(te+y)$ is real rooted (in t) $\quad \forall y \in \mathbb{R}^d$.*

There are non-homogeneous generalizations that we will not consider for now. One question to ask is what is the set of all directions such that p is hyperbolic in those directions. One reason is the following result.

Theorem 1.2. *p is a real stable polynomial and p has no roots in the positive orthant iff homogenization of p is hyperbolic in every e in the positive orthant. [Given $p, p_H(z_0, z_1, ..., z_n) = z_0^d p(\frac{z_1}{z_0}, ..., \frac{z_n}{z_1})$]*

Note: The restriction that p has no roots in the positive orthant is a purely technical restriction. This is because most of the time p is some polynomial like the generator of a distribution and has non-negative coefficients. Other times it is possible to shift p by a constant such that no root of p is in the positive orthant.

It is often easy to argue that p_H is hyprbolic in some specific direction. The question is then how do we go to proving the same for an entire orthant. To do this, we will need to prove some structural results on the set of directions in which p is hyperbolic. This involes a somewhat weird construction.

Definition 1.3. *Let p be a polynomial hyperbolic in some direction e. Then*

$$\kappa(p,e) = \{x \in \mathbb{R}^n : p(x-te) \text{ has positive roots}\}$$

Note: $e \in \kappa(p,e)$ as $p(e-te) = (1-t)^d p(e)$

Then we have the following theorem:

Theorem 1.4. *Let p be hyperbolic in direction e. Then*

1. *The set $\kappa(p,e)$ is the connected component of $\mathbb{R}^n \backslash \{x : p(n) = 0\}$*
2. *For any $v \in \kappa(p,e)$, p is hyperbolic in direction v.*
3. $\forall v \in \kappa(p,e), \kappa(p,e) = \kappa(p,v)$
4. *$\kappa(p,e)$ is convex.*

$\kappa(p,e)$ is called the hyperbolicity cone of p in direction e.

Before we look at a proof, let us quickly see why this could be useful.

Suppose p is a multivariate real polynomial with positive coefficients, then trivially p has no root in the positive orthant. Then by the theorem to prove that p is real stable it is enough to show that p is hyperbolic in direction $(1, ..., 1)$. As $\kappa(p, (1, ..., 1))$ must contain the positive orthant where there is no roots of p. Let's now take a look at the proof

Proof sketch. For (1):
Let C be some connected component of $R^n \backslash \{x : p(n) = 0\}$.

To show that $C \subseteq \kappa(p,e)$, we take $\tilde{e} \in C$. As C is connected, if a path $f : [0,1] \to C$ s.t. $f(0) = e, f(1) = \tilde{e}$, then the roots of $p(f(s) - te)$ continuously deform as s goes from 0 to 1. On the other hand,

$$f(s) \in C \implies p(f(s)) \neq 0 \implies p(f(s) - te)$$

never have roots at $t = 0$. Since all the roots at $s = 0$ are negative, all the roots at $s = 1$ are also negative. Therefore $C \subseteq \kappa(p,e)$

For the other side, as $\kappa(p,e)$ cannot contain roots of p, by maximality of C it is enough to show that $\kappa(p,e)$ is connected. Let $\tilde{e} \in \kappa(p,e)$. Then parameterize the line segment joining $\tilde{e}$ and e by $\frac{\tilde{e}+ce}{1+c}$ where $c \in [0,\infty)$. Then

$$p(\frac{\tilde{e}+ce}{1+c} - te) = \frac{1}{1+c} dp(\tilde{e} - (t(1+c) - c)e)$$

So for any t that is negative, $t(1+c) - c$ is also negative. Thus $p(\tilde{e} - (t(1+c) - c)e) \neq 0$ for $t < 0, c > 0$. Therefore

$$\frac{\tilde{e}+ce}{1+c} \in \kappa(p,e) \quad \forall c > 0$$

Hence the line between $\tilde{e}, e \in \kappa(p,e)$. Thus, $C = \kappa(p,e)$

This also implies (4), that $\kappa(p,e)$ is convex.

For (2), we will show that for any $\tilde{e} \in \kappa(p,e)$ and $\alpha, \beta > 0, x \in \mathbb{R}^n$, the roots of $p(\beta x - t\tilde{e} + i\alpha e)$ has all its roots in upper half plane $\mathcal{H}$. By taking $\alpha \to 0$ and $\beta = 1$, we will get that all roots of $p(x - t\tilde{e})$ are in $\mathcal{H}$. By conjugation, all roots of $p(x - t\tilde{e})$ are real.

This intuition for this idea is to treat p as a polynomial in two variables, one in the direction of $\tilde{e}$ and the other in e.

Consider the case where $\beta = 0$. Then

$$\begin{aligned} 0 &= p(-t\tilde{e} + i\alpha e) \\ &= (-t)^\alpha p\left(\tilde{e} - \frac{i\alpha}{t} e\right) \end{aligned}$$

Then as $\tilde{e} \in \kappa(p, e)$, $\frac{i\alpha}{t} > 0$. Hence, $\mathrm{Im}(t) > 0$.

Now for a general β, we will do a continuity argument. Suppose some β is such that $p(\beta x - t\tilde{e} - i\alpha e)$ has a root in $\mathcal{H}^C$. Then as at $\beta = 0$, $p(-t\tilde{e} + i\alpha e)$ has all roots in $\mathcal{H}$. There must be some β_0 by continuity such that one root of $p(\beta_0 x - t\tilde{e} + i\alpha e)$ is on the interface of $\mathcal{H}$ and $\mathcal{H}^C$. But as $\beta_0 x - t_0\tilde{e} \in \mathbb{R}^n$,

$p(\beta_0 x - t\tilde{e} + se)$ has no complex root as p is hyperbolic in e.

(3) follows from (1) and (2).

□

To get more intuition about the set $\kappa(p, e)$, consider the case when $p = \det(\sum z_i A_i))$ where A_i are symmetric and e s.t. $\sum e_i A_i = I$. Then $\kappa(p, e)$ i.e. x s.t. $p(x - te)$ has positive roots is $\{x : \sum x_i A_i$ is positive definite.$\}$

Thus if we were to view $\kappa(p, e)$ not as elements of $\mathbb{R}^n$ but as coordinates of elements in $p\{A_1, ..., A_n\}$ and thus elements in $M_{n\times n}, \kappa(p, e)$ could be viewed as the intersection of positive definite matrices with a hyperplane. The big question is if it is always true, bringing us to the following conjecture

Conjecture 1.5. *Let p be a homogeneous polynomial hyperbolic in e. Does there exist a g and matrices $A_1, ..., A_n$*

1. $fg = \det(z_1 A_1 + ... + z_n A_n)$
2. $\kappa_e(f) \subset \kappa_e(g)$
3. $\sum e_i A_i > 0$

Fundamentally this is of use to hyperbolic program, i.e. $\min_{x\in\kappa(p,e)} c^T x$. Lax conjecture says that it is the same as a SDP.

2 Open Problems

Finally, we will end off our series by introducing a open problems.

Problem 2.1. *Construct a series of $d-$regular graphs whose second non-trivial eigenvalue is exactly $2\sqrt{d-1}$.*

This problem is to construct a "rich" enough class of such graphs. Alternatively, if this is not possible, can one get second order bounds on the second eigenvalue thereby tightening the Alon-Bopanna bound of $2\sqrt{d-1} - 2\frac{2\sqrt{d-1}-1}{\mathrm{dist}(H)}$.

But a less silly are the following criticisms of the current construction of Ramanujan graphs. Bipartite graphs have two top eigenvalues. So in some sense it is never really Ramanujan, even though both the top two eigenvalues are considered trivial (as they have eigenvectors which are independent of the structure of the graph).

Problem 2.2. *For any $d \geq 2$, construct a sequence of $d-$regular graphs with increasing vertices such that the second largest eigenvalues in modulus is smaller than $2\sqrt{d-1}+O(1)$.*

The hardness of trying to directly use interlacing arguments since it allows us to control bounds in any one direction.

However one idea would be to somehow construct a bipartite graph from a non-bipartite graph such that if the bipartite one is Ramanujan, so is the non-bipartite. A somewhat related generalization comes from looking at a different matrix from graphs called the Hashimoto non-backtracking matrix B defined as follows.

For an undirected graph G, make it directed by doubling every edge. Then B is a $2m \times 2m$ matrix with

$$B_{e,f} \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } h(e) = t(f) \text{ and } e \neq f^{-1} \\ 0 & \text{otherwise} \end{cases}$$

For $d-$regular graphs, the Ihara-Bass formula relates eigenvalues of A to that of B. Thus one can propose a definition of Ramanujan for B, where G is $d-$regular.

Problem 2.3. *Prove a tight Alon-Bopanna type result on B. Construct sequences of raphs which satisfy tightness result.*

A starting point for this problem would be to link the roots of the expected characteristic polynomial of the non backtracking operators of the 2-lifts to the spectral radius of the same of the universal cover.

Problem 2.4. *Let h be some graph with T as universal cover. Let H be a uniform random 2-lift. Then*

$$|\operatorname{MaxRoot}\left[E_H[x(B_H)]\right]| \leq |\rho(B_T)|$$

where $\rho(B_T)$ is the spectral radius of B_T

The next problem is related to the concentration results.

Problem 2.5. *Under a reasonable model of generating random $d-$regular graphs of size n, can one show that with constant probability or at least "good" probability the graph generated is Ramanujan.*

In particular this asks if the existence of one element can be improved upon to give a count. There are multiple related problems in this vein.

For instance the technique used to get Kadison-Singer can be slightly modified to prove the following:

Theorem 2.6. *Given any graph G with n edges, there is a subset H with $\lceil dn \rceil$ edges such that*

$$L_G \leq L_H \leq \left(\frac{d+1+2\sqrt{d}}{d-1-2\sqrt{d}}\right) L_G = (1+\epsilon)L_G$$

Interestingly, a probabilistic version of this can be obtained by sampling in accordance to the effective resistance of each edge. However this methodology needs one to sample $O(n \log n)$ many edges and in fact give that with high probability one should have the resulting gloves to be a sparsifier. It is then reasonable to imagine that losing the $\log n$ should depreciate the w.h.p. to constant probability.

Problem 2.7. *Construct a random Algorithm that sparsifies G to have only $O(n)$ edges.*

Another direction of problems relate to proving hyperbolicity for certain weirder class of polynomials. For instances

Problem 2.8. *Show that if $L_1, ..., L_n$ are line segments. Then the Steiner polynomial given by*

$$p(z_1, ..., z_n) = \mathrm{vol}(\sum z_l L_i)$$

is hyperbolic, where $L_1 + L_2$ is the set of Minkowski sum.

This is then related to a $L_1 - s$ version of Weaver's conjecture which among other things prove Goddyn's conjecture, which in turn will allow a constant factor approximate solution to the travelling salesman problem.

In particular the Weaver's conjected which is proved says for any vectors v_i satisfying

$$\langle v_i, x\rangle^2 \leq \epsilon \sum_j \langle v_j, x\rangle^2$$

There is a partition $T_1 \cup T_2 = [n]$ such that

$$\sum_{i \in T_j} \langle v_i, x\rangle^2 \leq \left(\frac{1}{2} + O(\sqrt{\epsilon})\right) \sum_{i=1}^{m} \langle v_i, x\rangle^2$$

The open problem is

Problem 2.9. *Can one replace all the $\langle v_i, x\rangle^2$ terms by $|\langle v_i, x\rangle|$.*